Realizing Differential Graded Algebra Structures on Minimal Free Resolutions

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Summary

Ideals in the algebra of power series in three variables can be classified based on algebra structures on their minimal free resolutions. The classification is incomplete in that it remains open which algebra structures actually occur; this ***realizability question*** was formally raised by Avramov in 2012. We survey which classes have been realized in the literature and detail the presenter's contributions towards an answer for the realizability question.

Minimal Free Resolutions

A *free resolution* of ideal I in a ring R is a sequence of free R-modules

$$\mathbf{F}_\bullet:\quad \dots \longrightarrow F_3 \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} 0$$

such that $\operatorname{im}(d_{i+1}) = \ker(d_i)$ for $i \geq 1$ and $\operatorname{im}(d_1) = I$. A free resolution is *minimal* provided the rank of the free modules is the least possible.

Example 1. Consider the ring $R = k[\![x_1, x_2, x_3]\!]$ and the ideal $I = (\mathbf{x})$ where $\mathbf{x} = x_1, x_2, x_3$ is a regular sequence in R. The minimal free resolution of R/I below is called the Koszul resolution:

$$\begin{tikzcd}
& 0 \arrow[dr] & & 0 & & 0 \arrow[dr] & & 0 & & \\
& & K_2 \arrow[ur] \arrow[dr, "\iota_2"] & & & & I \arrow[ur] \arrow[dr, "\iota_0"] & & & \\
& R \arrow[ur, "\pi_3"] \arrow[rr, dotted, "d_3"] & & R^3 \arrow[dr, "\pi_2"] \arrow[rr, dotted, "d_2"] & & R^3 \arrow[ur, "\pi_1"] \arrow[rr, dotted, "d_1"] & & R \arrow[dr, "\pi_0"] & & \\
0 \arrow[ur, "\iota_4"] & & & & K_1 \arrow[ur, "\iota_1"] \arrow[dr] & & & & R/I \arrow[dr] & \\
& & & 0 \arrow[ur] & & 0 & & & & 0
\end{tikzcd}$$

where $d_3 = \begin{pmatrix} x_3 \\ -x_2 \\ x_1 \end{pmatrix}$, $d_2 = \begin{pmatrix} x_2 & x_3 & 0 \\ -x_1 & 0 & x_3 \\ 0 & -x_1 & -x_2 \end{pmatrix}$, and $d_1 = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$.

Classifying Resolutions of Length 3

Let I be a perfect ideal of grade 3 in a local ring R. Set $m = \operatorname{rank}_R(F_1)$ and $n = \operatorname{rank}_R(F_3)$ and write $\mathbf{F}_\bullet$ as

$$0 \longrightarrow R^n \xrightarrow{d_3} R^{m+n-1} \xrightarrow{d_2} R^m \xrightarrow{d_1} R \longrightarrow 0$$

We look at $\mathbf{A}_\bullet = \mathrm{H}(\mathbf{F}_\bullet \otimes_R k) = \operatorname{Tor}^R_\bullet(R/I, k)$ and consider the induced product on $\mathbf{A}_\bullet$. Choose bases

$$\{\mathsf{e}_i\}_{i=1,\dots,m}, \quad \{\mathsf{f}_i\}_{i=1,\dots,m+n-1}, \quad \{\mathsf{g}_i\}_{i=1,\dots,n}$$

of A_1, A_2, and A_3, respectively. Set $p = \dim A_1A_1$, $q = \dim A_1A_2$, and $r = \operatorname{rank}\delta_A$ for the natural homomorphism $\delta_A : A_2 \to \operatorname{Hom}_k(A_1, A_3)$ defined via $\delta_A(y)(x) = xy$. By results of [2], there are five distinct classes of multiplicative structures on $\mathbf{A}$:

$\mathbf{C}(3)$	$\mathsf{e}_1\mathsf{e}_2 = \mathsf{f}_3$, $\mathsf{e}_2\mathsf{e}_3 = \mathsf{f}_1$, $\mathsf{e}_3\mathsf{e}_1 = \mathsf{f}_2$	$\mathsf{e}_i\mathsf{f}_i = \mathsf{g}_1$ for $1 \leq i \leq 3$
$\mathbf{T}$	$\mathsf{e}_1\mathsf{e}_2 = \mathsf{f}_3$, $\mathsf{e}_2\mathsf{e}_3 = \mathsf{f}_1$, $\mathsf{e}_3\mathsf{e}_1 = \mathsf{f}_2$	
$\mathbf{B}$	$\mathsf{e}_1\mathsf{e}_2 = \mathsf{f}_3$	$\mathsf{e}_i\mathsf{f}_i = \mathsf{g}_1$ for $1 \leq i \leq 2$
$\mathbf{G}(r)$		$\mathsf{e}_i\mathsf{f}_i = \mathsf{g}_1$ for $1 \leq i \leq r$
$\mathbf{H}(p,q)$	$\mathsf{e}_i\mathsf{e}_{p+1} = \mathsf{f}_i$ for $1 \leq i \leq p$	$\mathsf{e}_{p+1}\mathsf{f}_{p+j} = \mathsf{g}_j$ for $1 \leq j \leq q$

Survey of Previous Results

Given the classification of algebra structures, the values of p, q, and r are fixed for classes $\mathbf{C}(3)$, $\mathbf{T}$, and $\mathbf{B}$. By results of [1] and [6], we have the following restrictions on p, q, and r for class $\mathbf{G}(r)$ and $\mathbf{H}(p,q)$:

$\mathbf{G}(r)$	$p = 0,$	$q = 1,$	$r \leq m, r \neq m-1$
$\mathbf{H}(p,q)$	$p \leq \min(m-1, n+1), p \neq n$	$q \leq \min(n, m-2), q \neq m-3$	$r = q$

The following tables visualize the possible classes with respect to their corresponding values of p, q, and r within the parameters $4 \leq m \leq 9$ and $2 \leq n \leq 9$, along with additional results towards the realizability question. The black boxes are classes realized in the literature (see [3], [4], [5], [6], [7], and [8]), the gray boxes are classes proved to be unrealizable, and the white boxes are possible classes that are not realized in the current literature.

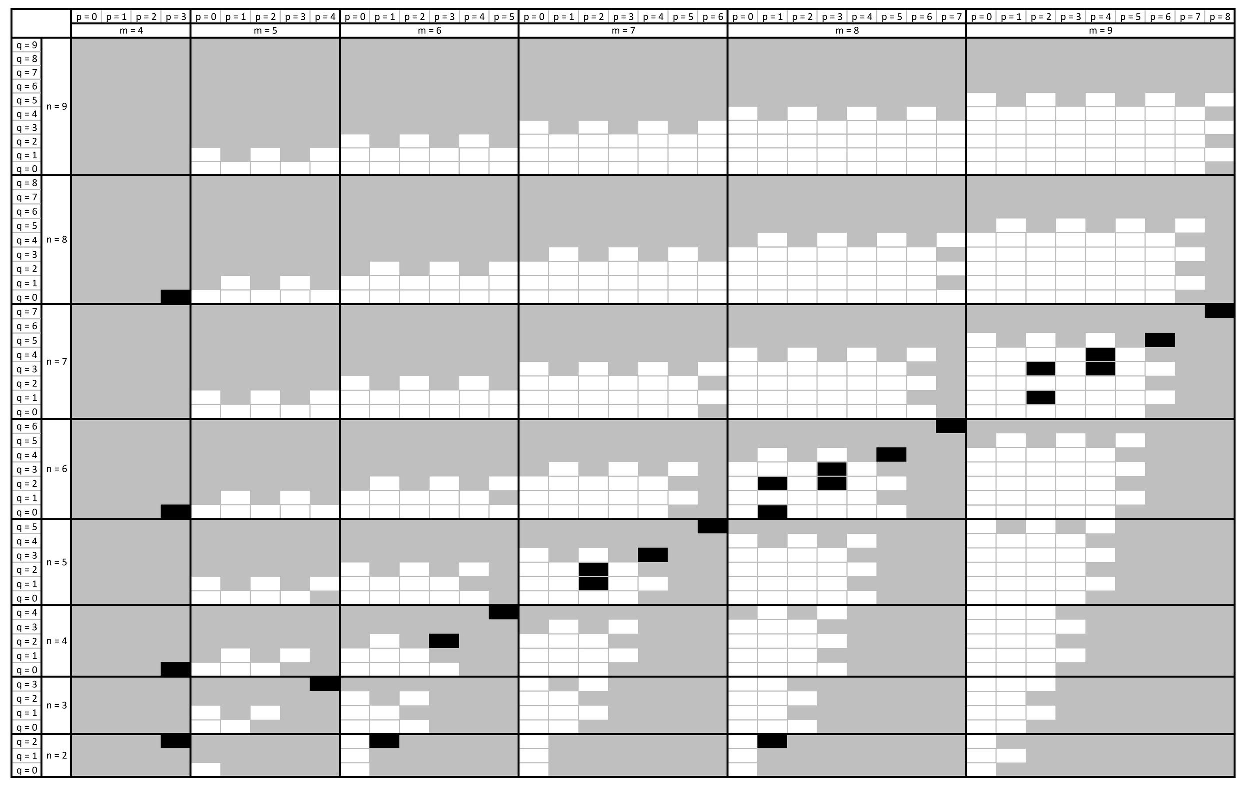

Fig. 1: Previously Constructed ideals of class $\mathbf{H}$

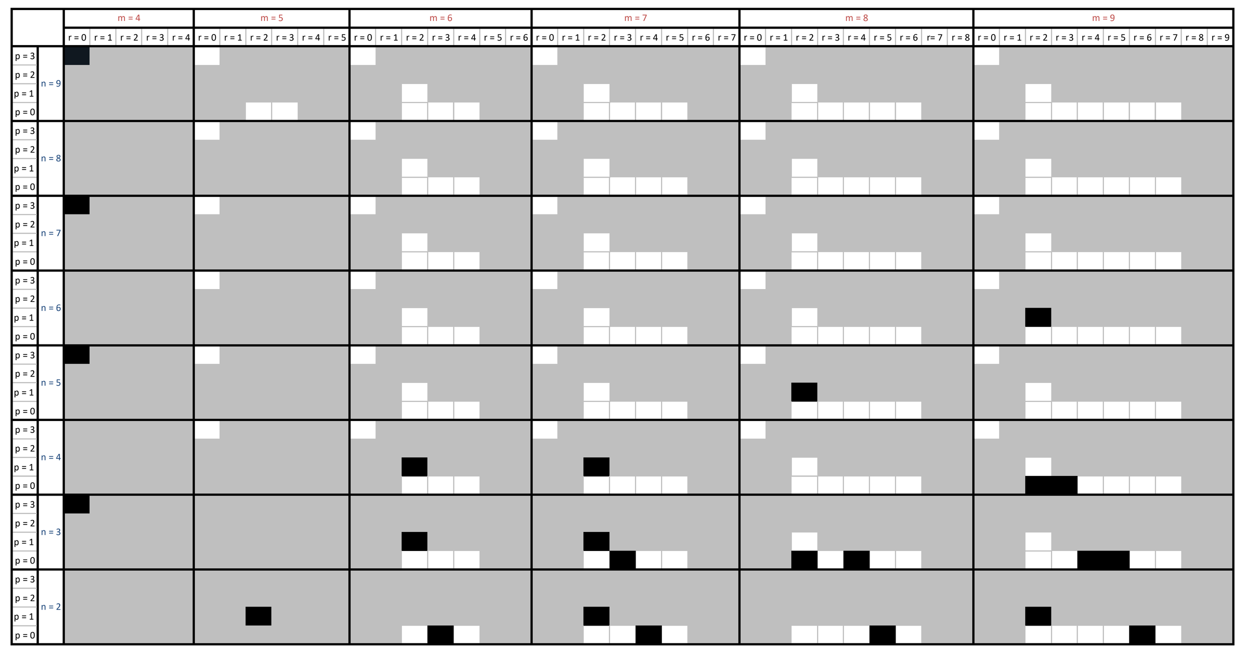

Fig. 2: Previously Constructed ideals of class $\mathbf{B}$, $\mathbf{G}(r)$, and $\mathbf{T}$

New Results

Let I be a perfect ideal of grade 3 and $\mathbf{x} = x_1, x_2, x_3$ a regular sequence in I. Consider the ideal $J = (\mathbf{x}) : I$, said to be ***linked to*** I ***by*** $\mathbf{x}$. By a result of [2], we can construct a map ψ as an extension of the map $\phi : R/\mathbf{x} \to R/I$ such that $\operatorname{cone}(\psi)$ is a free resolution of R/J. This process is called ***linkage*** and was used to obtain the original classification and new results.

Theorem 1. *For all $m \geq 5$ and $n \geq 4$, we can realize ideals of class $\mathbf{T}$.*

Theorem 2. *For all $m \geq 6$ and $n \geq 3$, we can realize ideals of class $\mathbf{B}$.*

Theorem 3. *For $m \geq 5$ and $n \geq 3$, we can realize ideals of all classes $\mathbf{H}(p,q)$ with $p = n-1$ and $q = m-4$ within the parameters proved in [1] and [6].*

In the tables below, the blue boxes are classes realized in the theorems stated above.

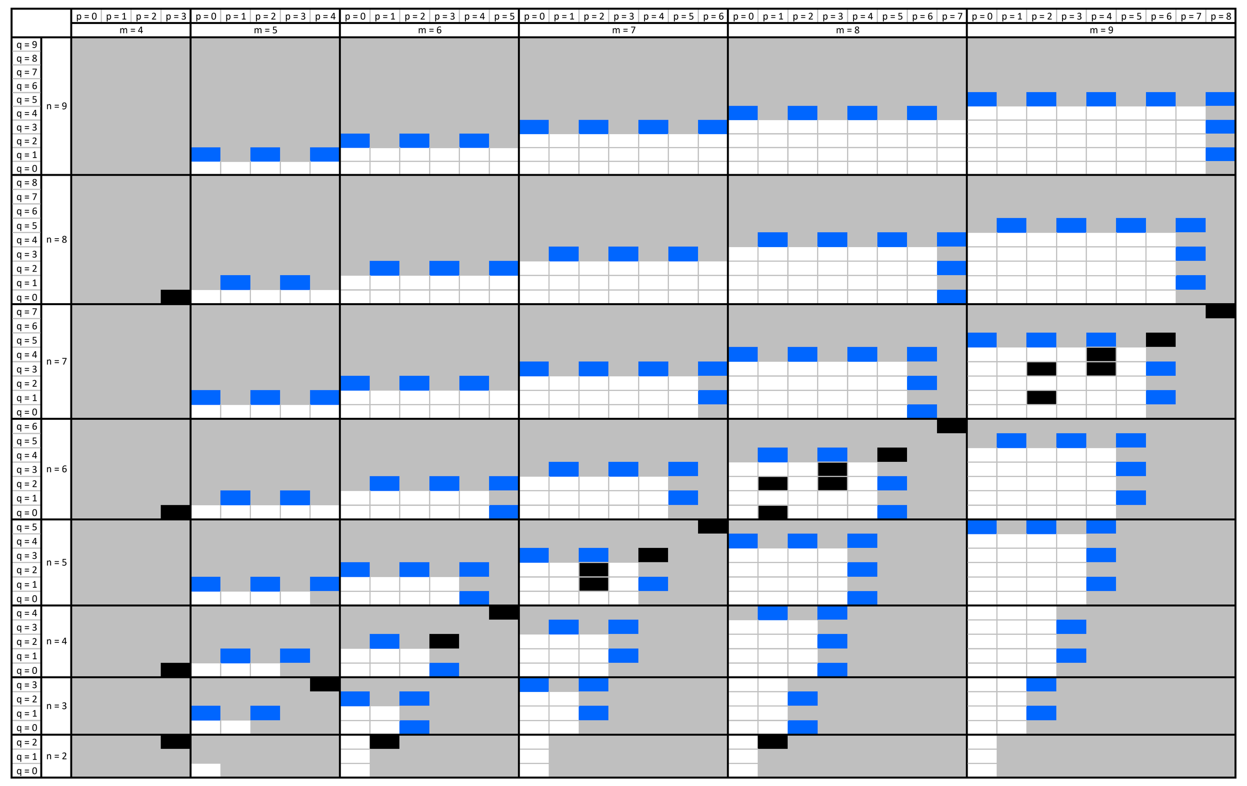

Fig. 3: Newly Constructed ideals of class $\mathbf{H}$

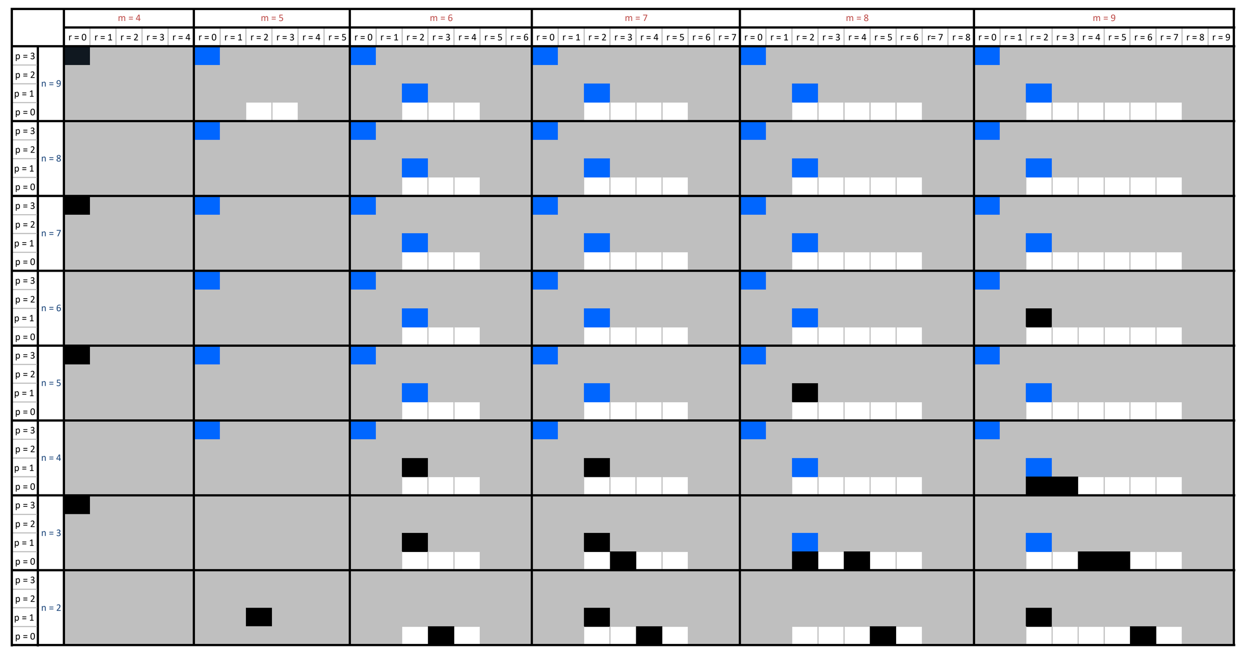

Fig. 4: Newly Constructed ideals of class $\mathbf{B}$, $\mathbf{G}(r)$, and $\mathbf{T}$

References

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