# Classifying Ideals Based on Multiplicative Structures in Homology


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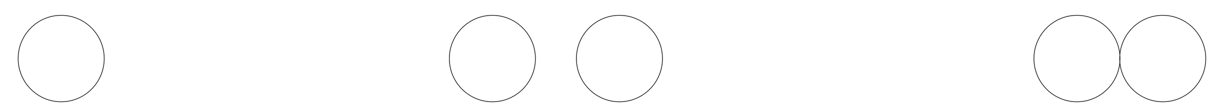
## Summary


Ideals in the algebra of power series in three variables can be classified based on the multiplicative structure on their Tor algebras. The classification is incomplete in that it remains open which structures actually occur; this realizability question was formally raised by Avramov in 2012. An answer to this question would give insight into possible generating functions for the Betti numbers of ideals in local rings, an important homological invariant. In this work, we survey which classes have been realized in the literature and detail the presenter's contributions towards an answer to the realizability question.


## Motivation

Equations define curves in space. For example, consider the curves defined by the following equations:

$x^2 + y^2 = 1$ $\qquad$ $(x-2)^2 + y^2 = 1$, $(x+2)^2 + y^2 = 1$ $\qquad$ $(x-1)^2 + y^2 = 1$, $(x+1)^2 + y^2 = 1$



The first curve has **1** component with **1** hole. The second curve has **2** components with **2** holes. The third curve has **1** component with **2** holes. As the equations defining curves become more complicated, we can count 0, 1, 2 and higher dimensional holes to understand what a space looks like. The number of holes in each dimension defines a *betti number* and can be observed from either a visualization from the space **or** from the equations defining the space.

Let $I$ be an ideal in a local ring $(R, \mathfrak{m}, k)$ generated by the defining equations of a space. We compute a *free resolution* of $Q = R/I$ as a sequence of free $R$-modules

$$\mathbf{F}_\bullet : \quad \dots \longrightarrow F_3 \xrightarrow{d_3} F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0 \xrightarrow{d_0} 0$$

such that $\operatorname{im}(d_{i+1}) = \ker(d_i)$ for $i \geq 1$ and $\operatorname{im}(d_1) = I$.

We look at the Tor algebra $\mathbf{A}_\bullet = \mathrm{H}(\mathbf{F}_\bullet \otimes_R k) = \operatorname{Tor}^R_\bullet(R/I, k)$. We define the $i^{th}$ *betti number* of $I$ as $\dim(A_i)$. Betti numbers are more often studied together in a *Poincaré series*, defined as

$$P_k^Q(t) = \sum_{i \geq 0} (-1)^i \dim(A_i)\, t^i$$

In the 1980's, it was an open question if one could write the Poincaré series as a a rational function. To contribute towards an answer for this question, Avramov, Kustin, and Miller studied rings with particular properties.

## The Classification

Let $I$ be a perfect ideal of grade $3$ in a regular local ring $R$. Set $m = \operatorname{rank}_R(F_1)$ and $n = \operatorname{rank}_R(F_3)$ and write $\mathbf{F}_\bullet$ as

$$0 \longrightarrow R^n \xrightarrow{d_3} R^{m+n-1} \xrightarrow{d_2} R^m \xrightarrow{d_1} R \longrightarrow 0$$

and consider the induced product on $\mathbf{A}_\bullet$. Choose bases $\{e_i\}_{i=1,\dots,m}$, $\{f_i\}_{i=1,\dots,m+n-1}$, $\{g_i\}_{i=1,\dots,n}$ of $A_1$, $A_2$, and $A_3$, respectively. Set $p = \dim A_1 A_1$, $q = \dim A_1 A_2$, and $r = \operatorname{rank} \delta_A$ for the natural homomorphism $\delta_A : A_2 \to \operatorname{Hom}_k(A_1, A_3)$ defined via $\delta_A(y)(x) = xy$. By results of [2], there are five distinct classes of multiplicative structures on $\mathbf{A}$:

| | | |
|---|---|---|
| $\mathbf{C}(3)$ | $e_1e_2 = f_3,\ e_2e_3 = f_1,\ e_3e_1 = f_2$ | $e_if_i = g_1$ for $1 \leq i \leq 3$ |
| $\mathbf{T}$ | $e_1e_2 = f_3,\ e_2e_3 = f_1,\ e_3e_1 = f_2$ | |
| $\mathbf{B}$ | $e_1e_2 = f_3$ | $e_if_i = g_1$ for $1 \leq i \leq 2$ |
| $\mathbf{G}(r)$ | | $e_if_i = g_1$ for $1 \leq i \leq r$ |
| $\mathbf{H}(p,q)$ | $e_ie_{p+1} = f_i$ for $1 \leq i \leq p$ | $e_{p+1}f_{p+j} = g_j$ for $1 \leq j \leq q$ |

By [1, Theorem 2.1], the Poincaré series has the form $\frac{(1+t)^2}{g(t)}$, where $g(t)$ is defined for each class:

| | |
|---|---|
| $\mathbf{C}(3)$ | $(1-t)^3(1+t)^2$ |
| $\mathbf{T}$ | $1 - t - (m-1)t^2 - (n-3)t^3 - t^5$ |
| $\mathbf{B}$ | $1 - t - (m-1)t^2 - (n-1)t^3 - t^4$ |
| $\mathbf{G}(r)$ | $1 - t - (m-1)t^2 - nt^3 + t^4$ |
| $\mathbf{H}(p,q)$ | $1 - t - (m-1)t^2 - (n-p)t^3 + qt^4$ |

Determining which values of $m$ and $n$ are possible for each class helps to determine which Poincaré series, and therefore, which geometric spaces, are possible. Realizing specific classes with specific values of $m$ and $n$ has become known as the *realizability question*. A sample of results towards this question from [1, 3, 4, 5] is below:

- If an ideal has class $\mathbf{T}$, then one must have $m \geq 4$ and $n \geq 3$. Moreover, if $m = 4$, then $n \geq 3$ is odd and if $m \geq 5$, then $n \geq 4$.
- If an ideal has class $\mathbf{B}$, the one must have $m \geq 5$ and $n \geq 2$. Moreover, if $m = 5$, then $n = 2$ and if $n = 2$, then $m$ is odd.
- If an ideal has class $\mathbf{H}(p,q)$, the one must have $m \geq 4$ and $n \geq 2$. Moreover,
  - If $m = 4$ and $n = 2$, then $p = 3$ and $q = 2$.
  - If $m = 4$ and $n \geq 3$, then $n$ is even, $p = 3$, and $q = 0$.
  - If $m \geq 5$, $n = 2$, and $p \geq 1$, then $m$ is even, $p = 1$, and $q = 2$.
  - If $p = n - 1$, then $q \equiv_2 m - 4$ and if $q = m - 4$, then $p \equiv_2 n - 1$.

## Realizing Classes

As one may notice, the results to the left restrict the values of $m$ and $n$ for each class. Another approach to the realizability question is constructing examples of classes for particular $m$ and $n$ values, which is done in the results below.

**Theorem 1.** *For all $m \geq 4$ and $n \geq 3$, one can realize ideals with class $\mathbf{T}$.*

**Theorem 2.** *For all $m \geq 5$ and $n \geq 2$, one can realize ideals with class $\mathbf{B}$.*

**Theorem 3.** *For $m \geq 4$ and $n \geq 2$, one can realize ideals with class $\mathbf{H}(p,q)$ with $p = n - 1$ or $q = m - 4$ within the parameters proved in [1] and [5].*

The following tables visualize the possible classes with respect to their corresponding values of $p$, $q$, and $r$ within the parameters $4 \leq m \leq 10$ and $1 \leq n \leq 10$. The gray boxes are classes that are not realizable. The gold boxes are classes that have been realized. Gold boxes with horizontal stripes are those that were constructed in the literature prior to this work. The white boxes are classes for which the realizability question is still open.

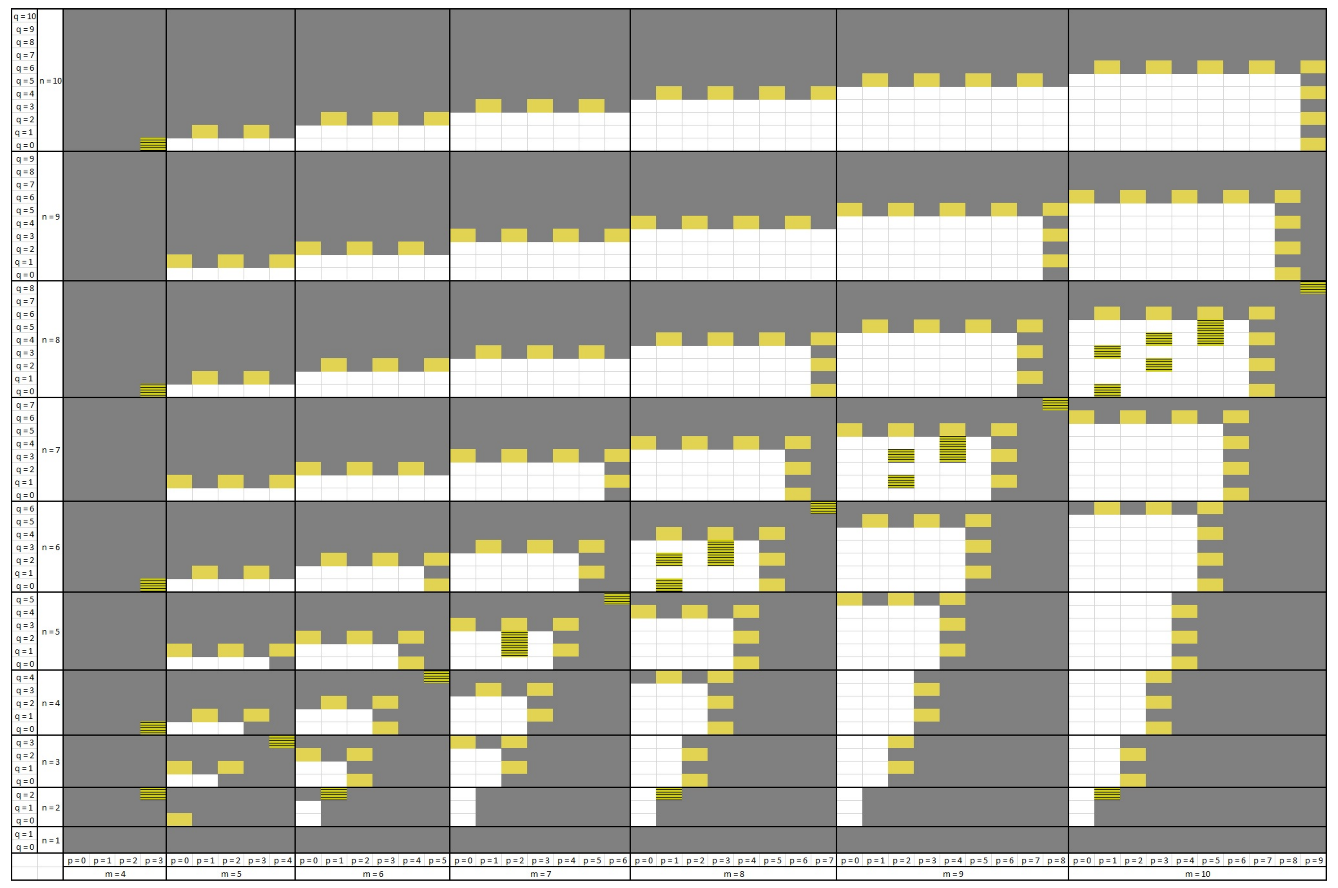


Fig. 1: Class $\mathbf{H}(p,q)$

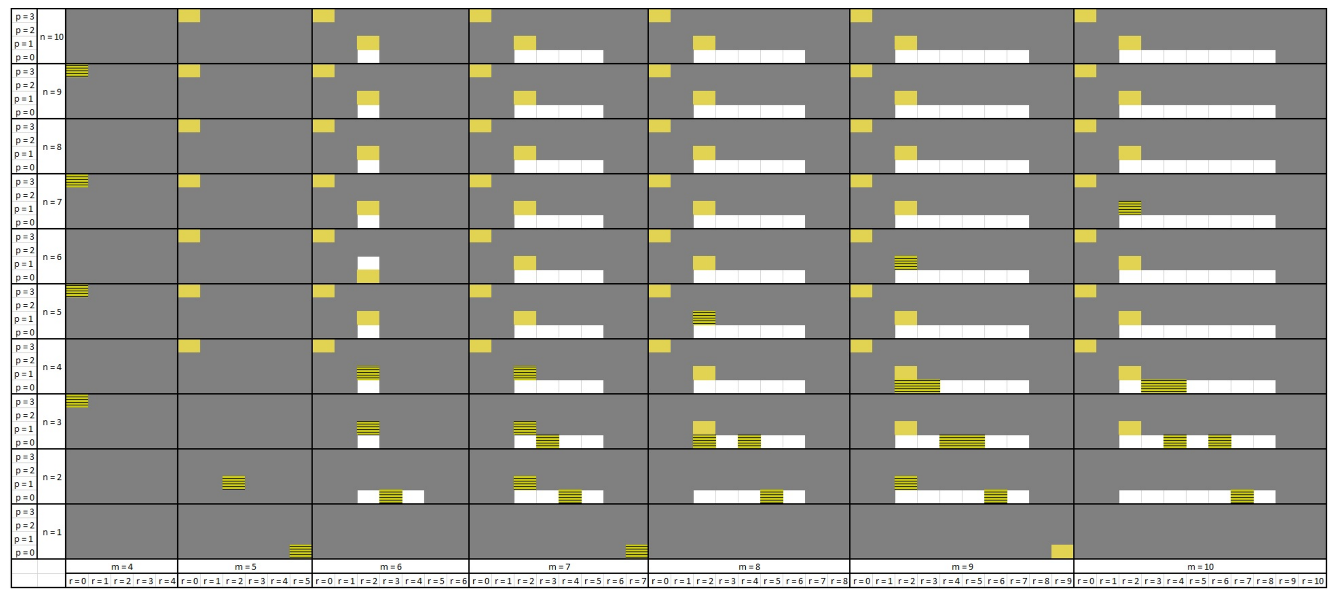


Fig. 2: Class $\mathbf{B}$, Class $\mathbf{G}(r)$, and Class $\mathbf{T}$

## References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