A DG algebra resolution of trimmings of Pfaffian ideals

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Abstract

Let $(R, \mathfrak{m}, \mathbf{k})$ be a regular local ring of dimension 3. Let I be a Gorenstein ideal of R of grade 3. Buchsbaum and Eisenbud[2] proved that there is a skew-symmetric matrix of odd size such that I is generated by the sub-maximal pfaffians of this matrix. Let J be the ideal obtained by multiplying some of the pfaffian generators of I by $\mathfrak{m}$; we say that J is a trimming of I. In this project we construct an explicit free resolution of R/J with a DG algebra structure. Our work builds upon a recent paper of Vandebogert [4]. We use our DG algebra resolution to prove that recent conjectures of Christensen, Veliche and Weyman [3] on trimmings of ideals of class **G** hold true in our context.

Background

Let I be a ideal of grade 3 with minimal free resolution F. A DG algebra structure on F. induces a graded $\mathbf{k}$-algebra structure on $\mathrm{Tor}^R(R/I, \mathbf{k}) \coloneqq \mathrm{H}.(F. \otimes_R \mathbf{k})$ that is unique and will belong to one of the following classes: $\mathbf{C}(3)$, $\mathbf{T}$, $\mathbf{B}$, $\mathbf{G}(r)$, $\mathbf{H}(p,q)$ whose products are described in the table below[1].

$\mathbf{C}(3)$	$e_1e_2 = f_3, e_2e_3 = f_1, e_3e_1 = f_2$	$e_if_i = g_1$ for $1 \le i \le 3$
$\mathbf{T}$	$e_1e_2 = f_3, e_2e_3 = f_1, e_3e_1 = f_2$	
$\mathbf{B}$	$e_1e_2 = f_3$	$e_if_i = g_1$ for $1 \le i \le 2$
$\mathbf{G}(r)$		$e_if_i = g_1$ for $1 \le i \le$ r
$\mathbf{H}(p,q)$	$e_ie_{p+1} = f_i$ for $1 \le i \le p$	$e_{p+1}f_{p+j} = g_j$ for $1 \le j \le q$

Let $T = (T_{i,j})$ be an $m \times m$ skew-symmetric matrix with entries in $\mathbf{k}$. We define a function $\mathcal{P}$ from the set of words in the letters $\{1, \dots, m\}$ to $\mathbf{k}$ as follows

$$\mathcal{P}[i,j] \coloneqq T_{i,j}, \quad \text{for } i,j \in \{1, \dots, m\}$$
$$\mathcal{P}[i_1 \dots i_n] \coloneqq 0, \qquad \text{if } n \text{ is odd}$$
$$\mathcal{P}[i_1 \dots i_n] \coloneqq \sum \mathrm{sgn}\begin{pmatrix} i_1 \dots i_{2k} \\ j_1 \cdots j_{2k} \end{pmatrix} \mathcal{P}[j_1j_2] \cdots \mathcal{P}[j_{2k-1}j_{2k}], \qquad \text{if } n \text{ is even}$$

where the sum is taken over all the partitions of $\{i_1, \dots, i_{2k}\}$ in k subsets of size 2. If $\{i_1, \dots, i_n\} \subseteq \{1, \dots, m\}$ with $i_1 < \cdots < i_n$, we define the pfaffians of the submatrix T as

$$\mathrm{pf}_{i_1 \dots i_n}(T) \coloneqq \mathcal{P}[i_1 \dots i_n]$$
$$\mathrm{pf}_{\overline{j_1 \dots j_n}}(T) \coloneqq \mathrm{pf}_{\{1,\dots,\mathrm{m}\} \setminus \{j_1 \dots j_n\}}(T)$$

Let I be the ideal generated by $y_i = (-1)^{i+1}\mathrm{pf}_{\bar{i}}(T)$ for $i = 1, \dots, m$. A product on a minimal free resolution F. of I will be denoted by $- \cdot_F -$ and is given by

$$e_i \cdot_F e_j \coloneqq \sum_{r=1}^{m} (-1)^{i+1}\mathrm{sgn}\begin{pmatrix} (m)\setminus\{i\} \\ j, r(m)\setminus\{i,j,r\} \end{pmatrix} \mathrm{pf}_{\overline{i,j,r}}(T) f_r$$
$$e_i \cdot_F f_j \coloneqq \delta_{i,j} g \text{ [2]}$$

In particular, I is of class $\mathbf{G}(\mathrm{m})$. Let

$$J \coloneqq y_1\mathfrak{m} + \cdots y_t\mathfrak{m} + (y_{t+1}, \dots, y_m).$$

where t is an integer between 1 and m. We say that J is obtained from I by trimming the first t generators of I.

A Free Resolution

Let I be a Gorenstein ideal of grade 3 generated by $\left((-1)^{i+1}\mathrm{pf}_{\bar{i}}(T)\right)$ for $i = 1, \dots, m$ for some skew-symmetric matrix T of odd size m with entries in R and let J be the ideal obtained by trimming the first t generators of I. In 2021, Vandebogert[4] proved the existence of maps

$$q_1^k : F_2 \to G_1^k$$
$$q_2^k : F_3 \to G_2^k$$

such that a free resolution of R/J is given by the mapping cone of the morphism

$$\begin{array}{ccccccc} 0 & \longrightarrow & F_3 & \longrightarrow & F_2 & \longrightarrow & F_1' \\ \downarrow & & \downarrow \begin{pmatrix} q_2^1 \\ \cdots \\ q_2^t \end{pmatrix} & & \downarrow \begin{pmatrix} q_1^1 \\ \cdots \\ q_1^t \end{pmatrix} & & \downarrow d_1' \\ \bigoplus_{k=1}^t G_3^k & \longrightarrow & \bigoplus_{k=1}^t G_2^k & \longrightarrow & \bigoplus_{k=1}^t G_1^k & \longrightarrow & R \end{array}$$

We can write this mapping cone as the sequence

$$0 \longrightarrow F_3 \oplus \left(\bigoplus_{k=1}^t G_3^k\right) \xrightarrow{\partial_3} F_2 \oplus \left(\bigoplus_{k=1}^t G_2^k\right) \xrightarrow{\partial_2} F_1' \oplus \left(\bigoplus_{k=1}^t G_1^k\right) \xrightarrow{\partial_1} R \longrightarrow 0$$

where the differentials are given by the matrices

$$\partial_3 = \begin{pmatrix} d_3 & 0 \\ \begin{pmatrix} q_2^1 \\ \dots q_2^t \end{pmatrix} & \bigoplus_{k=1}^t \delta_3^k \end{pmatrix} \quad \partial_2 = \begin{pmatrix} d_2' & 0 \\ -\begin{pmatrix} q_1^1 \\ \dots q_1^t \end{pmatrix} & \bigoplus_{k=1}^t \delta_2^k \end{pmatrix} \quad \partial_1 = \begin{pmatrix} d_1 & -\sum_{k=1}^t y_k \delta_1^k \end{pmatrix}$$

Let $l, s, \alpha, \beta \in \{1,2,3\}$ with $\alpha < \beta$, let k be an integer between 1 and t, and let i, j be integers between 1 and m. We define the constants

$$\sigma_{i,j,r} \coloneqq (-1)^{i+1}\mathrm{sgn}\begin{pmatrix} (m)\setminus\{i\} \\ j, r(m)\setminus\{i,j,r\} \end{pmatrix}$$
$$\sigma_{i,j,r,h,k} \coloneqq \mathrm{sgn}\begin{pmatrix} (m)\setminus\{i,j,r\} \\ kh(m)\setminus\{i,j,r,k,h\} \end{pmatrix}$$
$$d_{\alpha,\beta}^k \coloneqq \sum_{i=1}^{m}\sum_{r=1}^{m} \sigma_{i,k,r} c_{i,k,\beta} c_{r,k,\alpha} \mathrm{pf}_{\overline{i,j,r}}(T)$$
$$d_{\alpha,\beta}^{k,i,j} \coloneqq \sum_{r=1}^{m} \sigma_{i,j,r} \sum_{h=1}^{m} \sigma_{i,j,r,h,k} \mathrm{pf}_{\overline{i,j,r,h,k}}(T) c_{r,k,\beta}\, c_{h,k,\alpha}$$

One can verify that defining the maps q_1^k and q_2^k by

$$q_1^k(f_i) = \sum_{l=1}^{3} c_{i,k,l}\, u_l^k,$$
$$q_2^k(g) \coloneqq d_{1,2}^k v_{1,2}^k + d_{1,3}^k v_{1,3}^k + d_{2,3}^k v_{2,3}^k$$

guarantees the commutativity of the following diagrams and therefore define a free resolution as given above.

$$\begin{array}{ccc} F_3 & \longrightarrow & F_2 \\ q_1^k \downarrow & & q_2^k \downarrow \\ G_2^k & \longrightarrow & G_1^k \end{array} \qquad \begin{array}{ccc} F_2 & & \\ q_1^k \downarrow & \searrow D_0^{k'} & \\ G_2^k & \xrightarrow{\delta_1^k} & \mathfrak{m} \end{array}$$

A DG Algebra Structure

Let G^k be the Koszul resolution of $\mathbf{k}$ over R. The DG algebra structure on G^k is the usual exterior product of the Koszul complex, denoted as $- \cdot_{G^k} -$.

Theorem 1. A DG algebra structure on the resolution of $R/J \otimes_R \mathbf{k}$ is given by the following products:

a) $e_i \cdot e_j := e_i \cdot_F e_j + \sum_{k=1}^t d_{1,2}^{k,i,j}\, v_{1,2}^k + d_{1,3}^{k,i,j} v_{1,3}^k + d_{2,3}^{k,i,j} v_{2,3}^k$

b) $e_i \cdot f_j \coloneqq e_i \cdot_F f_j$ if $t + 1 \le j \le m$

c) $e_i \cdot f_j \coloneqq -\sum_{r=1}^m c_{r,j,3} d_{1,2}^{j,i,r} w^j$ if $1 \le j \le t$

Given the work of Vandebogert[3], to verify that these products do indeed give a DG algebra strucure, one need only verify that the Leibniz rule is satisfied.

We denote by Q_1 the transpose of the matrix $(q_1^1, \cdots, q_1^t)$. We denote by $p(T,t)$ the number of pivot columns of $Q_1 \otimes_R \mathbf{k}$ among the last $m - t$ columns. We denote by a bar the residue class modulo $\mathfrak{m}$.

Theorem 2. The trimmed ideal J is of format

$$(1, m + 2t - \mathrm{rank}(Q_1 \otimes_R \mathbf{k}), m + 3t - \mathrm{rank}(Q_1 \otimes_R \mathbf{k}), 1 + t)$$

Moreover, if we denote the classes $\mathbf{H}(0,0)$ and $\mathbf{H}(0,1)$ by $\mathbf{G}(0)$ and $\mathbf{G}(1)$, respectively, then

1) If $m = 5$, then J is of class **G** if and only if the following condition holds: for every i, j, k distinct with $t + 1 \le i, j \le m$ and $1 \le k \le t$, set $\{r, h\} = [5]\setminus\{k, i, j\}$. The 2×2 minors of following the matrix are zero.

$$\begin{pmatrix} \overline{c_{h,k,1}} & \overline{c_{h,k,2}} & \overline{c_{h,k,3}} \\ \overline{c_{r,k,1}} & \overline{c_{r,k,2}} & \overline{c_{r,k,3}} \end{pmatrix}$$

2) If $m \ge 7$, then J is always of class **G**.

Furthermore, if J is of class **G**, then $r = m - t - p(T,t)$.

To see the format of J, one must consider the ranks of the free modules in the resolution and remove basis elements corresponding to the units in Q_1. To see that J is of class **G** when $m = 5$, one must notice that product (a) is zero precisely when the condition listed is satisfied. When $m \ge 7$, one must notice that product (a) is always zero. To see the value of r, one must consider how creating a minimal resolution changes the basis and when $m = 5$, consider the two cases $p(T,t) = m - t$ and $p(T,t) < m - t$.

Applications

Corollary. Let $\mu(J)$ denote the minimal number of generators of J, then

1) If $t = 1$, then $r = \mu(J) - 3$.
2) If $t \ge 2$, then $r \le \mu(J) - 4$.

In addition to some algebraic manipulation, the proof relies on two facts:

1) If $t = 1$, then $\mathrm{rank}(Q_1 \otimes_R \mathbf{k}) = p(T,t)$.
2) If $t \ge 2$, then $\mathrm{rank}(Q_1 \otimes_R \mathbf{k}) - p(T,t) \le t$.

This corollary confirms the conjectures given by Christensen, Veliche and Weyman[3] when trimming the pfaffian generators of a Gorenstein ideal.

Example. Consider the ideal I of class $\mathbf{G}(7)$ generated by sub-maximal pfaffians of the skew-symmetric matrix

$$T = \begin{pmatrix} 0 & x & 0 & 0 & 0 & x & z \\ -x & 0 & 0 & 0 & x & z & y \\ 0 & 0 & 0 & x & z & y & 0 \\ 0 & 0 & -x & 0 & y & 0 & 0 \\ 0 & -x & -z & -y & 0 & 0 & 0 \\ -x & -z & -y & 0 & 0 & 0 & 0 \\ -z & -y & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

1) If $t = 1$, then $\mathrm{rank}(Q_1 \otimes_R \mathbf{k}) = 2$, $p(T,1) = 2$, and J is of class $\mathbf{G}(4)$ with 7 minimal generators.
2) If $t = 2$, then $\mathrm{rank}(Q_1 \otimes_R \mathbf{k}) = 4$, $p(T,2) = 2$, and J is of class $\mathbf{G}(3)$ with 7 minimal generators.

These examples show that the bounds in the corollary above are indeed realized, which contributes to solving the realizability question for class $\mathbf{G}(r)$: given the minimal number of generators and the type of an ideal, which values of r can be realized by some ideal of codimension 3?

Future Work

In this work, we generated many formulas to aid in our calculations involving pfaffians. These formulas could be applied to other classes of ideals, such as almost complete intersections, whose products can also be stated in terms of pfaffians. Trimming these ideals may contribute to answering the realizability question for other classes.

References

1) Luchezar L. Avramov, Andrew R. Kustin, and Matthew Miller, Poincare series of modules over local rings of small embedding codepth or small linking number, J. Algebra 118 (1988), no. 1, 162–204. MR 961334
2) David A. Buchsbaum and David Eisenbud, Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3, Amer. J. Math. 99 (1977), no. 3, 447–485. MR 453723
3) Lars Winther Christensen, Oana Veliche, and Jerzy Weyman, Linkage classes of grade 3 perfect ideals, J. Pure Appl. Algebra 224 (2020), no. 6, 106185, 29. MR 4048506.
4) Keller Vandebogert, DG structure on length 3 trimming complexes and applications to tor algebras, 2021.